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Application of Geometric Decoupling Theory to Synthesis of Aircraft Lateral Control Systems

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A vector space formulation is used to develop a computational procedure for decoupling a multivariable system. The concepts fundamental to this geometric approach are defined and their relation to the behavior of the linear system is discussed. Two theorems that provide the basis for the procedure are stated. The vector space statements are translated into algebraic operations suitable for machine calculation. A principal result of these computations is a set of linear algebraic equations that the elements of a decoupling feedback matrix must satisfy. The set of all solutions defines (in certain cases) the class of feedback matrices that will give the desired decoupling. A penalty function method is used to choose a matrix from this class so that the augmented system has desirable handling qualities. Two examples from the open literature are used to illustrate the method.

Introduction

IN the crowded terminal areas of today's air transportation system it would be helpful both to provide more precise flight-path control and to reduce pilot workload. These two improvements, which are often conflicting, can be achieved by providing a decoupled flight control system. The core of the decoupling concept is that each control should affect a single output without disturbing others. Pilot workload is reduced since coordination of controls is no longer required. Additionally, flight-path precision can be ehanced since the pilot can focus his attention on a particular output.

The decoupling problem has received considerable attention and several approaches are possible. One method, perhaps the most obvious one, is restricted to the case when the outputs to be decoupled are scalars (e.g., decoupling bank angle from sideslip). The germ of the idea is to find feedback and cross control matrices such that the output/input transfer frunction is diagonal and nonsingular. Using such an approach Falb and Wolovich[1] and Gilbert[2] have established methods of solving this decoupling problem. Falb and Wolovich[1] also present a synthesis procedure which in some cases provides a means of achieving a specified pole-placement while simultaneously decoupling the system. Gilbert and Pivnichny[3] have also reported some computational procedures based on their approach to the synthesis problem. Other computational methods have also been used. Model following techniques were employed by Yore[4] and by Hall.[5] Additionally, Montgomery and Hatch[6] have applied their general synthesis method to the decoupling problem. On the theoretical side an elegant approach, based on a vector space formulation, was developed independently by Wonham and Morse,[7–9] and Basile and Marro.[10, 11] The computational procedures used here are based, in part, on this geometric approach and this is, to our knowledge, the first reported application of this geometric theory.

Our approach to the combined problems of decoupling and handling qualities proceeds in two stages. First we find (in certain cases) the most general class of feedback matrices F and cross control matrices G that lead to a decoupled system. We then use a quadratic penalty function method in conjunction with a numerical descent algorithm to arrive at (or close to) specified handling qualities.

The first step, that of finding the most general F and G matrices, follows the geometric approach of Ref. 9. The theory is outlined in the next section, which includes a simple example. An essential part of the calculation procedure is explained in Appendix A. The following section describes the penalty function method used in specifying handling qualities. We then consider two examples dealing with aircraft lateral control systems. The final section includes a discussion of some difficulties and comparisons with other methods.

Geometric Theory of Decoupling

In order to analyze aircraft motions the standard procedure, which will be followed here, is to linearize the equations of motion about some nominal flight condition. If this condition is a steady motion then the resulting equations are

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linear and time invariant. The decoupling problem considered is for systems of this type.

A linear time-invariant dynamical system may be described by

$$\dot{x} = Ax + Bu \tag{1}$$

where x is the n-component state vector, u is the m-component control vector and A and B are constant matrices of the appropriate sizes. Additionally, let there be an output equation

$$y = Hx \tag{2}$$

which we choose to decompose into subvectors

$$y_i = H_i x, \qquad i = 1, \ldots, k \tag{3}$$

Here each y_i is a vector of dimension q_i so that the sum of the q_i is the dimension of y in Eq. (2). The y_i are grouped as physically meaningful variables.

We are concerned with the problem of finding a feedback law

$$u = U(x, u^c) \tag{4}$$

so that when Eq. (4) is used in Eq. (1) the system has both a decoupled structure, and suitable response characteristics. Loosely, we want u^c (command inputs) to be composed of subvectors u_i^c ($i = 1, 2, \ldots, k$) so that the subvector u_i^c affects only the output subvector y_i and causes no change in the others. Response characteristics (handling qualities) will be specified in terms of frequency domain behavior of the system.

Since the original system is linear we seek a special form of Eq. (4), namely

$$u = Fx + \sum_{i=1}^{k} G^i u_i^c$$

or

$$u = Fx + Gu^c$$

where

$$G = [G^1, G^2, \ldots, G^k]$$

and

$$u^c = [u_1^c, u_2^c, \ldots, u_k^c]^T$$

Thus, the augmented system is

$$\dot{x} = (A + BF)x + BGu^c$$

In order to discuss the geometric approach some vector space concepts must be introduced. It is assumed that the reader is familiar with vector space notions of subspaces, bases, invariance and orthogonality and with the Gram-Schmidt procedure. The text by Halmos[12] is a good source of this material.

Invariant Subspaces

Definition: A subspace $\mathbf{V} \subset \mathbf{X}$‡ is an invariant subspace of (A, B) if there is a matrix F such that $\mathbf{V}$ is invariant under $(A + BF)$. This notion relates to the decoupling problem in the following way. Suppose the system is started with a particular output (say y_i) zeroed. It is of interest to know if that output will stay zero. If the null space of the output contains an invariant subspace of (A, B) (say $\mathbf{V}$) the question is partially answered. That is, we know there is a feedback matrix F such that if the system is in $\mathbf{V}$ at some time and the only control used is feedback $F \cdot x$ then the system will stay in $\mathbf{V}$ (so that y_i stays zero).

It is clear then, that given the null space of the other outputs we would like to compute an invariant subspace of (A, B) contained in this null space. Furthermore it would be nice if we could find the largest invariant subspace of (A, B) contained in this null space. This leads to Theorem 1. *Theorem 1*: Let $\mathbf{N} \subset \mathbf{X}$ be a fixed subspace, then there is a subspace $\mathbf{V}^M \subset \mathbf{N}$ such that $\mathbf{V}^M$ is the maximal invariant subspace of (A, B) contained in $\mathbf{N}$. Furthermore, $\mathbf{V}^M$ is given by $\mathbf{V}_\mu$ where $\mu = \dim(\mathbf{N})$ and the $\mathbf{V}_i$ are defined recursively§ by

$$\mathbf{V}_0 = \mathbf{N};\ \mathbf{V}_i = \mathbf{V}_{i-1} \cap A^{-1}(\mathbf{V}_{i-1} + \mathbf{B}),\ i = 1, 2, \ldots, \mu.$$

Proof: This theorem is a composition of theorems and lemmas from Sec. 3 of Ref. 7.

The invariant subspace plays an important role in the development of the decoupling theory. Note, however, that Theorem 1 makes no mention of a feedback matrix F that renders $\mathbf{V}^M$ invariant under $(A + BF)$. Happily, given A, B, and $\mathbf{V}^M$, it is not difficult to derive conditions that F must satisfy in order that $\mathbf{V}^M$ be an invariant subspace of (A, B). The required conditions, which are essential to our approach, are derived in Appendix A. At this stage we only point out that the theorem guarantees the existence of such an F. Let us denote by $\tilde{F}(\mathbf{V})$ the class of feedback matrices which render $\mathbf{V}$ invariant under $(A + BF)$.

Obviously, the invariant subspaces do not provide a complete answer to the decoupling problem. External control of the system does not enter, so it is not known if attempts to control one output will cause the zeroed outputs to change.

Controllability Subspaces

Before proceeding with the second (and last) concept it is convenient to introduce some notation. Given the $(n \times n)$ matrix A and a subspace $\mathbf{B} \subset \mathbf{E}^n$ define the subspace

$$\{A/\mathbf{B}\} = \mathbf{B} + A\mathbf{B} + \cdots + A^{n-1}\mathbf{B}$$

A proof that $\{A/\mathbf{B}\}$ is a subspace and a discussion of its properties can be found in Ref. 13. For our purpose it is sufficient to point out that $\{A/\mathbf{B}\}$ is the set of points that the system (1) can reach from the origin using arbitrary controls.

Definition: A subspace $\mathbf{R} \subset \mathbf{X}$ is a controllability subspace of (A, B) if there is a matrix F such that

$$\mathbf{R} = \{A + BF/\mathbf{B} \cap \mathbf{R}\}$$

From the definition of $\mathbf{R}$ and the Hamilton-Cayley theorem it is not difficult to show that a controllability subspace of (A, B) is also an invariant subspace of (A, B). The controllability subspaces play a central role in the geometric approach to decoupling. Before stating some results we shall discuss the concept and its relation to decoupling.

Suppose $\mathbf{R}_i$ is a controllability subspace. Let the intersection of $\mathbf{B}$ and $\mathbf{R}_i$ be spanned by a set of vectors $\{r_1^i, \ldots, r_s^i\}$. Define the $(m \times s)$ matrix G^i so that

$$BG^i = [r_1^i, \ldots, r_s^i]$$

then the controllability subspace is precisely the set of points that can be reached from the origin by the system

$$\dot{x} = (A + BF)x + BG^i u_i^c$$

where u_i^c takes vaues in $\mathbf{E}^s$.

Thus, if we could find a controllability subspace (say $\mathbf{R}_i$) in the null space of the other output matrices, then starting from any point in $\mathbf{R}_i$ we could use arbitrary controls u_i^c and still

‡ Upper case letters in boldface denote subspaces, which are stored in the computer as matrices. For example, $\mathbf{B}$ is the subspace spanned by the columns of the matrix B.

§ The sequence of subspaces ($\mathbf{V}_i$) is nested (monotone nonincreasing) and is defined recursively. Thus, if the dimension does not decrease at some stage then the subspaces have stopped changing and there is no point in continuing to $\mathbf{V}_\mu$. This observation is useful in performing the computations.

keep the other outputs at zero. The existence and characterization of a certain maximal controllability subspace is the subject of Theorem 2. *Theorem 2:* Let $\mathbf{V}$ be an invariant subspace of (A, B) then there is a largest controllability subspace (say $\mathbf{R}^M$) contained in $\mathbf{V}$ and it is given by $\mathbf{R}^M = \mathbf{S}_n$ where

$$\mathbf{S}_1 = \mathbf{B} \cap \mathbf{V} \quad \text{and} \quad \mathbf{S}_i = (A\mathbf{S}_{i-1} + \mathbf{B}) \cap \mathbf{V}$$

Proof: This theorem is a composition of several theorems and lemmas from Sec. 4, Ref. 7.

Precise Problem Statement

Utilizing the concept of a controllability subspace the decoupling problem may be stated somewhat more precisely. The system A, B, H_i $(i = 1, \ldots, k)$ is given. For each output matrix H_i compute the null space, say $\mathbf{N}_i$. We seek a set of controllability subspaces $\mathbf{R}_i$ such that

$$\mathbf{R}_i \subset \bigcap_{\substack{\tau=1 \\ \tau \neq i}}^{k} \mathbf{N}_\tau \qquad \text{(A)}$$

$$\mathbf{R}_i + \mathbf{N}_i = \mathbf{X} \qquad \text{(B)}$$

$$\bigcap_{i=1}^{k} \tilde{F}(\mathbf{R}_i) \neq \varnothing \qquad \text{(C)}$$

Condition (A) gives the desired noninteraction. Condition (B) guarantees that the control u_i^c can control completely the output y_i. Lastly, condition (C) insures that this can be accomplished with the same feedback matrix F.

An Overview of the Computations

With the matrices (A, B) and the null spaces $\mathbf{N}_i$ known we proceed as follows. For each i $(i = 1, \ldots k)$ find the intersection of the remaining $(k - 1)$ null spaces. Use the construction of Theorem 1 to compute the largest invariant subspace contained in this intersection. Next, make use of the algorithm of Theorem 2 to find the largest controllability subspace contained in the computed invariant subspace. For that "i" condition (A) is met, while condition (B) is easily checked. If $\mathbf{R}_i^M$ is too small the problem has no solution.

If condition (B) is satisfied we compute the intersection of $\mathbf{R}_i^M$ and $\mathbf{B}$ so that G^i can be computed. Lastly, we use the procedure outlined in Appendix A to construct a set of linear equations that the elements of F must satisfy in order that $\mathbf{R}_i^M$ be an invariant subspace. In general, it may happen that the class of feedback matrices, given by condition (C), is empty (i.e., the derived equations have no solution). It is natural then to seek controllability subspaces that are less than maximal (i.e., $\mathbf{R}_i \subset \mathbf{R}_i^M$, $i = 1, \ldots k$). Obviously condition (A) will be fulfilled. Condition (B) is not difficult to check and if all goes well the linear equations for F will have a solution. Unfortunately, at this time there is no simple procedure for finding a compatible set of $\{\mathbf{R}_i\}$ when $\{\mathbf{R}_i^M\}$ is not a compatible set.

Happily, in certain special cases of interest the $\{\mathbf{R}_i^M\}$ furnish the only possible solution. Thus, in these cases if $\{\mathbf{R}_i^M\}$ is not compatible then the problem has no solution. One such special case is when the B matrix has k linearly independent columns. That is, there are as many independent open loop controls as there are outputs to be decoupled. Proofs of this, as well as some other special cases, are given in Ref. 7. In what follows we are dealing with the aforementioned special case.

Since the $\{\mathbf{R}_i^M\}$ furnish the only (assuming there is one) solution to our problem then the general solution of the linear equations for F gives the most general decoupling feedback matrix for the problem. Thus, any decoupling feedback matrix F can be written as

$$F = F_o + \sum_{i=1}^{p} \mu_i F_i$$

where p is the dimension of the solution space for the homogeneous problem; the F_i provide a basis for this subspace; and the μ_i are arbitrary constants. F_o is any particular solution.

Each G^i is found as outlined above and G is the augmented matrix

$$G = [G^1, G^2, \ldots G^k]$$

It should be clear from the definition of the G^i that each column of G may be multiplied by any nonzero constant without affecting the results. Similarly, interchanging two columns of G merely reflects a relabeling of command inputs.

A Simple Example

We close the decoupling discussion by considering a simple example for which the required calculations can be done by hand

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H_1 = [0 \quad 0 \quad 1] \qquad H_2 = [1 \quad 1 \quad 0]$$

$\mathbf{N}_1$, the null space of H_1, is given by

$$\mathbf{N}_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

The largest invariant subspace in $\mathbf{N}_1$ is computed via the algorithm of Theorem 1

$$\mathbf{V}_0 = \mathbf{N}_1 \quad \text{and} \quad \mu = \dim(\mathbf{N}_1) = 2$$

We have

$$\mathbf{V}_1 = \mathbf{V}_0 \cap A^{-1}(\mathbf{V}_0 + \mathbf{B})$$

where

$$\mathbf{V}_0 + \mathbf{B} = \mathbf{E}^3$$

so that

$$A^{-1}(\mathbf{V}_0 + \mathbf{B}) = \mathbf{E}^3$$

and

$$\mathbf{V}_1 = \mathbf{V}_0$$

Thus (see the footnote following Theorem 1),

$$\mathbf{V}^M = \mathbf{V}_2 = \mathbf{V}_0 = \mathbf{N}_1$$

that is, the whole null space of H_1 is itself an invariant subspace of (A, B).

$\mathbf{R}_2^M$ is computed from the algorithm of Theorem 2. We have

$$\mathbf{S}_1 = \mathbf{B} \cap \mathbf{V}^M = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

To compute $\mathbf{S}_2$ we need

$$A\mathbf{S}_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \right\}$$

$$A\mathbf{S}_1 + \mathbf{B} = \mathbf{E}^3$$

so that,

$$\mathbf{S}_2 = \mathbf{V}^M \cap (A\mathbf{S}_1 + \mathbf{B})$$

or

$$\mathbf{S}_2 = \mathbf{V}^M \cap \mathbf{E}^3 = \mathbf{V}^M$$

Similarly we find

$$\mathbf{R}_2{}^M = \mathbf{S}_3 = \mathbf{S}_2 = \mathbf{V}^M = \mathbf{N}_1$$

With $\mathbf{R}_2{}^M$ known the procedure of Appendix A may be used to compute the class $\tilde{F}(\mathbf{R}_2{}^M)$ of decoupling feedback matrices. This is done in Appendix A.

To compute G^2 we need $\mathbf{B} \cap \mathbf{R}_2{}^M$

$$\mathbf{B} \cap \mathbf{R}_2{}^M = \operatorname{span}\{r_1\} = \operatorname{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

The condition

$$BG^2 = r_1$$

leads to

$$G^2 = [\lambda, o]^T$$

for any nonzero λ.

For output 2 the calculations give

$$\mathbf{N}_2 = \operatorname{span}\left\{\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}, \quad \mathbf{V}^M = \mathbf{N}_2$$

$$\mathbf{R}_1{}^M = \operatorname{span}\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}, \quad \mathbf{B} \cap \mathbf{R}_1{}^M = \mathbf{R}_1{}^M$$

and

$$G^1 = [0, \lambda]^T$$

for any nonzero λ.

The class of decoupling feedback matrices is computed to be

$$\begin{bmatrix} f_{11} & f_{12} & 0 \\ -1 & -1 & f_{23} \end{bmatrix}$$

with f_{11}, f_{12}, and f_{23} arbitrary (cf Ref. 1, p. 656).

Synthesis of Handling Qualities

Unfortunately, the results and procedures of this section are not as sharp or precise as the decoupling results. For this reason our approach is quite heuristic, although it is hoped the points are clear.

Given a decoupling pair (F, G) we can compute the input-state vector transfer function

$$\hat{x}(s)/\hat{u}^c(s) = [sI - (A + BF)]^{-1}BG$$

The most difficult part of these calculations is finding $[sI - (A + BF)]^{-1}$. This inverse is given by

$$[sI - (A + BF)]^{-1} = N(s)/d(s)$$

where

$$N(s) = N_1 s^{n-1} + \cdots + N_n$$

and

$$d(s) = s^n + d_1 s^{n-1} + \cdots + d_n$$

The matrices $N_1, \ldots, N_n$ and the scalars $d_1, \ldots, d_n$ may be computed from Leverrier's algorithm.[14]

The problem we are considering here is that of finding F and G so that certain of the elements of the matrices $N_1, \ldots N_n$, and the scalars $d_1 \ldots d_n$ have prescribed values.¶ In general the problem is quite difficult. Disregarding theoretical questions of existence, uniqueness and sensitivity the computational problem alone is not simple. For n greater than two, developing a closed form expression for the N_i and d_i in terms of F and G would be very tedious. Thus, the direct application of a root finding method, such as Newton's method, is not practical. Montgomery and Hatch[6] have successfully used a differential form of Leverrier's algorithm in developing a computational procedure for this problem.

Our approach is somewhat more gross and obviates (at some cost) the problem of solvability. Precisely, we use a penalty function method and seek to minimize a quadratic function of the error vector. The error is defined as the difference between the actual parameters and their respective desired values,

$$e = P_{\text{actual}} - P_{\text{desired}}$$

so the "cost" is $e^t Q e$. Q is selected so that the converged solution is sufficiently close to the desired system. The minimization is performed by a conjugate gradient algorithm (Ref. 15, pp. 294–297) employing a one-dimensional search by golden section.

¶ Actually we may specify that some function of these elements take a prescribed value. This is done in the first example of the next section.

Application to Aircraft Lateral Control Systems

In order to illustrate the method for more realistic problems we shall apply it to aircraft lateral motion control systems. Two models that have been examined elsewhere will be considered.

Montgomery's Aircraft[6]

This is a model with four state variables (p, ϕ, r, β) and two control variables $(\delta a, \delta r)$. The A and B matrices are given by

$$A = \begin{bmatrix} -0.367984 & 0.0 & -0.032279 & 26.18750 \\ 1.0 & 0.0 & 0.267949 & 0.0 \\ -0.024209 & 0.0 & -0.110395 & 4.46294 \\ 0.258819 & 0.017835 & -0.965926 & -0.091072 \end{bmatrix}$$

$$B = \begin{bmatrix} -7.67183 & 2.06549 \\ 0.0 & 0.0 \\ 1.96959 & -2.33843 \\ 0.0 & 0.0^{**} \end{bmatrix}$$

We seek to decouple the ϕ response and the β response so that (say)

$$H_1 = [0 \quad 1 \quad 0 \quad 0]$$

and

$$H_2 = [0 \quad 0 \quad 0 \quad 1]$$

The decoupling calculations were performed on an IBM 370/155 computer using a Fortran IV program with double precision arithmetic. The essential features of the computational procedures are discussed in Appendix B. Some of the results of the computations are given below (execution time was less than 2 sec).

$$\mathbf{R}_1{}^M = \operatorname{span}\left\{\begin{pmatrix}0.965937\\-0.004777\\0.258734\\0.0\end{pmatrix}, \begin{pmatrix}0.0\\0.999830\\0.018461\\0.0\end{pmatrix}\right\}$$

$$\mathbf{R}_2{}^M = \operatorname{span}\left\{\begin{pmatrix}0.258819\\0.0\\-0.965926\\0.0\end{pmatrix}, \begin{pmatrix}0.0\\0.0\\0.0\\1.0\end{pmatrix}\right\}$$

** Example 4 of Ref. 12 has $Cy_{\delta r} = 0$.

$$F_0 = \begin{bmatrix} 0.00825 & 0.0 & 0.0 & 0.0 \\ 0.03088 & 0.0 & -0.03689 & 17.36848 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} 0.56589 & -0.00054 & 0.13318 & 0.0 \\ 0.80417 & 0.00039 & 0.12384 & 0.0 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 0.0 & 0.58349 & 0.00207 & 0.0 \\ -0.00185 & 0.81206 & 0.00979 & 0.0 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} 0.0 & 0.0 & 0.20356 & 0.0 \\ -0.18184 & -0.01253 & 0.96195 & 0.0 \end{bmatrix}$$

$$F_4 = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.19745 \\ 0.0 & 0.0 & 0.0 & 0.98031 \end{bmatrix}$$

$$G = \begin{bmatrix} 1.0 & 0.20141 \\ 1.39192 & 1.0 \end{bmatrix}$$

In order to provide a proper comparison it was decided to choose the weighting constants for the F_i matrices ($i = 1, 4$) so that the augmented system had characteristic polynomial

$$(S + 0.01)(S + 1.5)(S^2 + 0.9S + 9)$$

and the numerator of the $\phi/\delta a$ transfer function was in the form

$$A(S^2 + 0.9S + 9.0)$$

These are the parameters selected by Montgomery and Hatch in their investigation (Example 4, Ref. 6).

The penalty function minimization approach discussed previously was used. Again a Fortran IV program was executed on an IBM 370/155 computer using double precision arithmetic. The weighting matrix Q is

$$Q = \text{diag}[1000.0, 1.0, 1.0, 50.0, 200.0, 2.0]$$

and the intial weights were

$$\mu = [-1.0, -1.0, -1.0, -1.0].$$

The method converged to

$$\mu = [0.3474, -0.0028, 0.4437, -23.52]$$

in approximately fifty descent cycles (each cycle consisting of one steepest descent followed by four conjugate direction searches). The feedback matrix is

$$F = \begin{bmatrix} 0.205 & 0.001 & 0.137 & -4.64 \\ 0.230 & -0.003 & 0.433 & -5.69 \end{bmatrix}$$

while Montgomery and Hatch give (cf Example 4, Ref. 6)

$$F = \begin{bmatrix} 0.205 & 0.002 & 0.135 & -4.64 \\ 0.230 & -0.003 & 0.430 & -5.69 \end{bmatrix}$$

The percentage error is quite small in all but the F_{12} term where the error magnitude is small. The cross control matrices agree at least to three digits. Simulation results do not differ noticably from those in Ref. 6. For this reason not such results are included here.

Hall's Aircraft[5]

The second aircraft is a model of a modified T-33 discussed by Hall.[5] The state variables are $x = (p, \phi, r, \beta)$ while the control variables are $(\delta a, \delta r, \delta p)$. δp is a yaw control being an asymmetric deflection of drag petals mounted on wing tip tanks. The A and B matrices are

$$A = \begin{bmatrix} -3.18 & 0.0 & 0.63 & -10.6 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ -0.06 & 0.0 & -0.27 & 4.18 \\ 0.022 & 0.0644 & -0.998 & -0.151 \end{bmatrix}$$

$$B = \begin{bmatrix} -14.4 & 1.5 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & -2.59 & -0.96 \\ 0.0 & 0.037 & 0.0 \end{bmatrix}$$

Following Hall we decouple p and r and β, thus

$$H_1 = [1, 0, 0, 0]$$
$$H_2 = [0, 0, 1, 0]$$
$$H_3 = [0, 0, 0, 1]$$

The decoupling program gives

$$\mathbf{R}_1{}^M = \text{span}\left\{ \begin{pmatrix} 1.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}, \begin{pmatrix} 0.0 \\ 1.0 \\ 0.0 \\ 0.0 \end{pmatrix} \right\}$$

$$\mathbf{R}_2{}^M = \text{span}\left\{ \begin{pmatrix} 0.0 \\ 0.0 \\ 1.0 \\ 0.0 \end{pmatrix} \right\}$$

$$\mathbf{R}_3{}^M = \text{span}\left\{ \begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 1.0 \end{pmatrix} \right\}$$

$$F_0 = \begin{bmatrix} 0.0 & 0.0 & 2.85343 & -0.73611 \\ -0.59459 & -1.74054 & 26.97297 & 0.0 \\ 1.54167 & 4.69583 & 0.0 & 4.35417 \end{bmatrix}$$

The nonzero elements of the F_i ($i = 1, 4$) matrices are given in Table 1, while the control interconnect matrix is given below.

$$G = \begin{bmatrix} 1.0 & 0.0 & 0.10416 \\ 0.0 & 0.0 & 1.0 \\ 0.0 & 1.0 & 2.69792 \end{bmatrix}$$

The freedom in the selection of a decoupling F was used to place the system poles so that the characteristic polynomial of the augmented system would be (cf Ref. 5)

$$S(S + 0.151)(S + 0.27)(S + 3.18)$$

The penalty function minimization program was used to obtain the feedback matrix

$$F = \begin{bmatrix} -0.2639 & -0.1813 & 2.8534 & -0.7347 \\ -0.5946 & 1.7405 & 26.973 & 0.0139 \\ 1.5417 & 4.6958 & -69.750 & 4.3167 \end{bmatrix}$$

which results in characteristic polynomial

$$S(S + 0.150)\,(S + 0.272)\,(S + 3.170)$$

Although the eigenvalues are close to the prescribed values, the feedback matrix differs from Hall's in the (1, 1) and (3, 4) elements (Hall[5] gives -0.062 and -73.0, respectively). The remaining elements show close agreement. In fact, due to poor accuracy, the values given by Hall do not lead to the desired pole placement.

Table 1 Nonzero elements of the feedback matrices

Matrix	(I, J)	Value
F_1	(1, 1)	1.0
F_2	(1, 2)	1.0
F_3	(3, 3)	1.0
F_4	(1, 4)	0.03618
F_4	(2, 4)	0.34732
F_4	(3, 4)	−0.93705

Discussion

For the problem considered the geometric approach to decoupling provides a simple constructive procedure to determine the classes of feedback and cross-control matrices that give a decoupled structure. The procedure has been translated into algebraic operations which are readily adapted to machine calculations. Concerning these calculations one numerical difficulty requires note. In the Gram-Schmidt orthonormalization procedure it is necessary to subtract from each new vector the projections of the preceding vectors on this vector. If the new vector is dependent upon the previous ones the result of this repeated subtraction is the zero vector. Thus, it is necessary to recognize when a vector is zero, often a problem in machine calculations. If it is not done correctly erroneous results may be obtained.

A related theoretical concern is that, as given, decoupling is an all or nothing condition. This means that a form of ill conditioning is likely wherein sensitivity to system parameters is a problem. It would be helpful to have a precise notion of nearly decoupled systems. The above mentioned computational difficulty suggests a possible foundation for such a notion. This is being investigated presently.

Since there are other computational approaches to the decoupling problem we shall briefly mention and comment on these competing methods. In solving his problem, Hall[5] used a model following approach. He specified a model that was decoupled and had desirable dynamic characteristics, and then chose a controller so that the original system would follow the model. This is a simple and clever approach; unfortunately, it does not always work. In fact, the controller used by Hall minimizes a certain norm (see Ref. 15, p. 103) but need not make the norm zero. Thus, perfect model following is not assured. For Hall's Aircraft if the tangent of the reference pitch angle is not zero ($\dot{\phi} \neq p$) the method fails.

As applied to the decoupling problem, the approach taken by Montgomery and Hatch[6] is without theory. They specified that the required elements in the output/input transfer function be zero and use a clever computational method to solve this problem.

Briefly their method involves writing the nonlinear algebraic equations in differential form and then integrating the set of implicit differential equations for the elements of F and G. Since the equations are implicit it is necessary to evaluate a gradient matrix at each integration step. For the decoupling aspects our approach yields a single set of linear algebraic equations and so is better both theoretically and practically. For the complete synthesis problem, it seems that a hybrid approach combining the decoupling procedure used here and handling qualities specification via their procedure might be quite efficient.

The computational schemes used by Gilbert and Pivnichny,[3] and Falb and Wolovich[1] are applicable only to the case of scalar outputs (i.e., the dimension of each y_i is unity). In most such cases either of these schemes provides a better synthesis procedure than our method because of the relative ease of solution of the pole placement problem. On the other hand our approach is more flexible allowing vector outputs and more general handling quality specifications.

Lastly, we remark that the algorithms given here are based only on part of the geometric theory of decoupling. For example, Ref. 9 has results on pole placement that have not been mentioned. The penalty function method was used because it allows more flexibility in handling quality specification. It is our opinion that for low-order systems it is a practical computational approach.

Appendix A: Calculation of Feedback Matrices

The point of this section is to indicate a method of computing the class of matrices $\tilde{F}(\mathbf{V})$ which render $\mathbf{V}$ (a given subspace) invariant under $(A+BF)$. This is the principal new result of this paper and makes the scheme practical. A different characterization of $\tilde{F}(\mathbf{V})$ is given in Ref. 9.

Let $B_v = \{v_1, \ldots, v_p\}$ be a basis for $\mathbf{V}$ and let $B_{vo} = \{v_{p+1}, \ldots, v_n\}$ be a basis for the orthogonal complement of $\mathbf{V}$. Any v in $\mathbf{V}$ can be written as a linear combination of the elements of B_v. Hence, in order that $\mathbf{V}$ be invariant under $A+BF$ it is necessary and sufficient that

$$(A+BF)v_i = \tilde{v}_i \varepsilon \mathbf{V}$$

for $i = 1, 2, \ldots, p$. Since $\tilde{v}_i$ is in $\mathbf{V}$, then for each fixed i we must have

$$[\tilde{v}_i/v_j] = 0 \qquad j = p+1, \ldots n \tag{A1}$$

Conversely, if Eq. (A1) applies then $\tilde{v}_i$ is in $\mathbf{V}$. Thus, if $\mathbf{V}$ is invariant under $(A+BF)$ then

$$[(A+BF)v_i/v_j] = 0 \qquad i = 1, \ldots p \qquad j = p+1, \ldots n \tag{A2}$$

and conversely. Equations (A2) can be written

$$[BFv_i/v_j] = -[Av_i/v_j] \qquad i = 1, \ldots p \qquad j = p+1, \ldots n \tag{A3}$$

Equations (A3) are $p(n-p)$ linear equations in the $m \cdot n$ unknown elements of F. The general solution gives $\tilde{F}(\mathbf{V})$. Before considering an example we make note of a technical point. It seems that the general solution of Eqs. (A3) may well depend upon which bases B_v and B_{vo} are used. It is not difficult to show that this is not the case.

Example: consider A, B as in the illustrative example.

$$B_v = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\right\} \quad \text{and} \quad B_{vo} = \left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

We have p equals two and n equals three. For $i = 1$ the calculations are

$$v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad Av_1 = \begin{pmatrix}0\\2\\1\end{pmatrix}, \quad BFv_1 = \begin{pmatrix}0\\f_{11}\\f_{21}\end{pmatrix}$$

j equals three is the only choice

$$v_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \quad [Av_1/v_3] = 1 \quad \text{and} \quad [BFv_1/v_3] = f_{21}$$

Thus the first equation is

$$f_{21} = -1$$

For the case $i = 2$ we have

$$v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad Av_2 = \begin{pmatrix}1\\3\\1\end{pmatrix}, \quad BFv_2 = \begin{pmatrix}0\\f_{12}\\f_{22}\end{pmatrix}$$

$$v_3 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad [Av_2/v_3] = 1, \quad [BFv_2/v_3] = f_{22}$$

so that

$$f_{22} = -1$$

The result is that any feedback matrix in the form

$$F = \begin{bmatrix}f_{11} & f_{12} & f_{13}\\-1 & -1 & f_{23}\end{bmatrix}$$

will have $\mathbf{V}$ invariant under $A+BF$. This is easily verified.

Appendix B: Algebraic Computations

In limited space it is not possible to give a detailed account of the computations. On the other hand since it may not be clear how to translate the vector space statements into algebraic computations some comment is in order.

The basic procedure used is the computation of an orthonormal set that spans the same subspace as a given set of vectors. The Gram-Schmidt procedure is a way of determining such an orthonormal set which is necessarily independent and thus provides a basis for the subspace.

To find the sum of two subspaces we form the set given by the union of the respective bases sets. We then compute an orthonormal set that spans the same subspace as the union. This new orthonormal set is a basis for the sum of the original subspaces.

A second vector space operation needed in implementing the decoupling algorithms is a way of finding the subspace of vectors orthogonal to a given subspace. If the original subspace is **V** we denote by $\mathbf{V}^{\perp}$ this orthogonal complement. Suppose we start with some basis for **V**. Add to this basis a basis for the whole n-dimensional space (the usual unit vectors (e_i) will do). Lastly, use the Gram-Schmidt process on this enlarged spanning set for $\mathbf{E}^n$. The nonzero vectors in the final basis that result from vectors in the added basis (the e_i) form a basis for $\mathbf{V}^{\perp}$.

Another required procedure is a method for finding the intersection of two subspaces. This can be reduced to applications of previous procedures by using the identity.

$$\mathbf{U} \cap \mathbf{V} = (\mathbf{U}^{\perp} + \mathbf{V}^{\perp})^{\perp}$$

The last procedure discussed is a method of finding the subspace of vectors that a given $(n \times n)$ matrix A will map into a given subspace (**V**) of $\mathbf{E}^n$. This is termed the inverse image of **V** under A and can be computed by

$$A^{-1}(\mathbf{V}) = (A^T\mathbf{V}^{\perp})^{\perp}$$

where A^T means the transpose of A. The theoretical foundations of these procedures may be found in the text by Halmos.[12]

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